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| Title | Complex Hadamard matrices coming from association schemes (Research on algebraic combinatorics, related groups and algebras) |
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$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix}.$$

J. Wallis, *Complex Hadamard matrices*,
Linear and Multilinear Algebra, 1 (3), (1973), 257–272.

entries: $\pm 1, \pm i$,

In this talk, we allow entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$.

A complex Hadamard matrix is said to be **Butson**-type, if all of its entries are roots of unity.

$\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$: commutative d -class association scheme.
 A_j : adjacency matrix $\longleftrightarrow R_j$.

Bose–Mesner algebra

- $\mathfrak{A} = \langle A_j \mid j = 0, \dots, d \rangle \subset M_n(\mathbb{C})$, Bose–Mesner algebra, semi-simple,

$$\mathfrak{A} = \langle A_j \mid j = 0, \dots, d \rangle = \langle E_j \mid j = 0, \dots, d \rangle,$$

$\{E_j\}_{j=0}^d$: the set of the primitive idempotents, $E_0 = \frac{1}{n}J$,

- $(A_0, A_1, \dots, A_d) = (E_0, E_1, \dots, E_d)P$,

$$P = \left(\begin{array}{c|ccc} 1 & k_1 & \cdots & k_d \\ \hline 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{array} \right),$$

P : the **first** eigenmatrix of \mathfrak{X} .

Our aim is to find both of

- a **complex** Hadamard matrix W
- an association scheme $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$

such that

$$W = A_0 + w_1 A_1 + \cdots + w_d A_d \in \mathfrak{A},$$

$$|w_1| = \cdots = |w_d| = 1.$$

Set

$$W = \sum_{j=0}^d w_j A_j \in \mathfrak{A}, \quad (1)$$

where $w_0 = 1$ and $|w_j| = 1$ for $\forall j \in \{1, \dots, d\}$. Then

$$W = \sum_{k=0}^d \left(\sum_{j=0}^d w_j P_{k,j} \right) E_k,$$

$$\overline{W}^\top = \sum_{j=0}^d \frac{1}{w_j} A_{j'} \quad (A_{j'} = A_j^\top)$$

$$= \sum_{k=0}^d \left(\sum_{j=0}^d \frac{1}{w_j} P_{k,j'} \right) E_k.$$

Takuya IKUTA Complex Hadamard matrices and association schemes

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E. R. van Dam, *Three-class association schemes*,
J. Algebraic Combin. 10 (1999), 69–107,
examples of symmetric 3 class association schemes = 103.

Appendix B

Four integral eigenvalues; excluded here are association schemes generated by $SRG \otimes J_n$,
and the rectangular schemes $R(m, n)$, except the 6-cycle C_6 and the Cube.

| v | spectrum | L_1 | L_2 | L_3 | # |
|-----|-----------------------------|-------|--------|-------|----------------------------------------------------|
| 6 | $\{2, 1^2, -1^2, -2^1\}$ | 0 1 0 | 1 0 1 | 0 1 0 | 1 $C_6 \cong R(3, 2)$ DRG Q-123 |
| | $\{2, -1, -1, 2\}$ | 1 0 1 | 0 1 0 | 1 0 0 | |
| | $\{1, -1, 1, -1\}$ | 0 2 0 | 2 0 0 | 0 0 0 | |
| v | spectrum | L_1 | L_2 | L_3 | # |
| 8 | $\{3, 1^3, -1^3, -3^1\}$ | 0 2 0 | 2 0 1 | 0 1 0 | 1 Cube $\cong R(4, 2)$ DRG Q-123 |
| | $\{3, -1, -1, 3\}$ | 2 0 1 | 0 2 0 | 1 0 0 | |
| | $\{1, -1, 1, -1\}$ | 0 3 0 | 3 0 0 | 0 0 0 | |
| 15 | $\{4, 2^3, -1^4, -2^3\}$ | 1 2 0 | 2 4 2 | 0 2 0 | 1 $L(\text{Petersen})$ DRG, R_1 SRG |
| | $\{8, -2, -2, 2\}$ | 1 2 1 | 2 4 1 | 1 1 0 | |
| | $\{2, -1, 2, -1\}$ | 0 4 0 | 4 4 0 | 0 0 1 | |
| 20 | $\{9, 3^3, -1^3, -3^3\}$ | 4 4 0 | 4 4 1 | 0 1 0 | 1 $J(6, 3)$ $R_1 \cong R_2$ DRG Q-123, Q-321 |
| | $\{9, -3, -1, 3\}$ | 4 4 1 | 4 4 0 | 1 0 0 | |
| | $\{1, -1, 1, -1\}$ | 0 9 0 | 9 0 0 | 0 0 0 | |
| 27 | $\{6, 3^4, 0^{12}, -3^4\}$ | 1 4 0 | 4 4 4 | 0 4 4 | 1 $R(3, 3)$ DRG Q-123 |
| | $\{12, 0, -3, 3\}$ | 2 2 2 | 2 5 4 | 2 4 2 | |
| | $\{8, -4, 2, -1\}$ | 0 3 3 | 3 6 3 | 3 3 1 | |
| 27 | $\{8, 2^{12}, -1^8, -4^4\}$ | 1 6 0 | 6 8 2 | 0 2 0 | 2 $Q(2, 4)^\text{spread}$ R_1 DRG, R_2 SRG |
| | $\{16, -2, -2, 4\}$ | 3 4 1 | 4 10 1 | 1 1 0 | |
| | $\{2, -1, 2, -1\}$ | 0 8 0 | 8 8 0 | 0 0 1 | |

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| | |
|---------|---------------------------------------------------------------------------------------------------------|
| | (symmetric) construction |
| $d = 3$ | <i>Complex Hadamard matrices contained in a Bose–Mesner algebra</i> , Spec. Matrices, 3 (2015), 91–110. |
| $d = 4$ | <i>Complex Hadamard matrices attached to even orthogonal scheme of class 4</i> , (2016), submitted. |

not *Butson*!

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$e \geq 3$: an *odd* positive integer

$$\text{GF}(2^e) = \frac{\text{GF}(2)[x]}{(\varphi(x))}, \quad \varphi(x) : \text{a primitive polynomial of degree } e \text{ over } \text{GF}(2)$$

$$\text{GF}(2^e)^\times = \langle \zeta \rangle$$

$$\mathbb{Z}_4 = \mathbb{Z} / 4\mathbb{Z}$$

$\exists \Phi(x) : \text{a monic polynomial of degree } e \text{ over } \mathbb{Z}_4 \text{ s.t.}$

$$\begin{cases} \Phi(x) \equiv \varphi(x) \pmod{2\mathbb{Z}_4[x]}, \\ \Phi(x) \mid x^{2^e-1} - 1 \text{ in } \mathbb{Z}_4[x]. \end{cases}$$

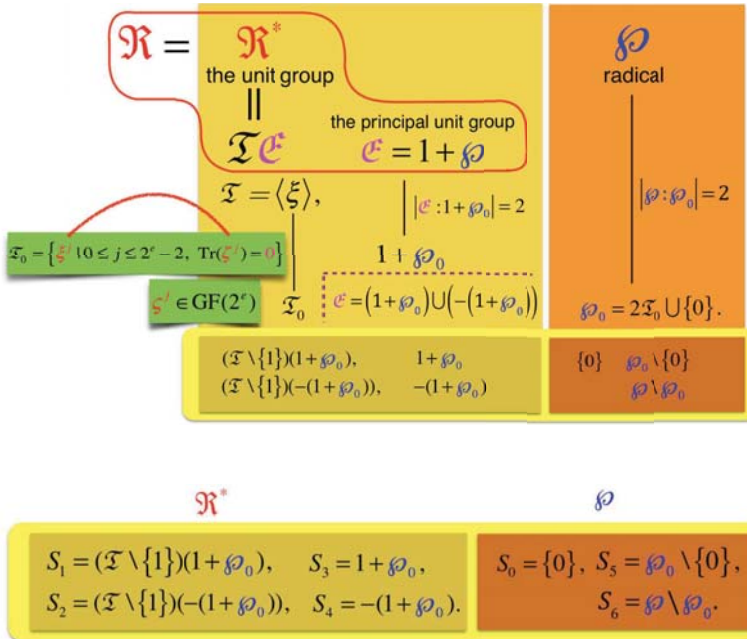
$$\mathfrak{R} = \frac{\mathbb{Z}_4[x]}{(\Phi(x))} : \text{Galois ring}$$

$$|\mathfrak{R}| = 4^e$$

$$\wp = 2\mathfrak{R}.$$

radical

$$|\wp| = 2^e$$



We've constructed a commutative nonsymmetric association scheme \mathfrak{X} of class 6 on Galois rings of characteristic 4, whose first eigenmatrix is given by

$$(p_{i,j})_{\substack{0 \leq i \leq 6 \\ 0 \leq j \leq 6}} = \begin{pmatrix} 1 & 2b(b-1) & 2b(b-1) & b & b & b-1 & b \\ 1 & bi & -bi & 0 & 0 & -1 & 0 \\ 1 & -bi & bi & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & bi & -bi & b-1 & -b \\ 1 & 0 & 0 & -bi & bi & b-1 & -b \\ 1 & -2b & -2b & b & b & b-1 & b \\ 1 & 0 & 0 & -b & -b & b-1 & b \end{pmatrix},$$

where b is a power of 4.

Theorem 2 (A. Munemasa and T. I.)

Let $w_0 = 1$ and w_j ($1 \leq j \leq 6$) be complex numbers of absolute value 1. Set

$$W = \sum_{j=0}^6 w_j A_j \in \mathfrak{A},$$

and assume that W is hermitian.

Then, W is a complex Hadamard matrix \iff

$$W = A_0 + \epsilon_1 i(A_1 - A_2) + \epsilon_2 i(A_3 - A_4) + A_5 + A_6, \quad \text{or} \quad (2)$$

$$W = A_0 + \epsilon_1 i(A_1 - A_2) + \epsilon_2(A_3 + A_4) + A_5 - A_6, \quad (3)$$

for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.

Let \mathfrak{X} be an association scheme given in Theorem 2. Fusion schemes of \mathfrak{X} with at least three classes are listed in Table 1.

| | fused relations | class | nonsymmetric or symmetric |
|------------------|--------------------------------|-------|---------------------------|
| \mathfrak{X}_1 | $\{1, 2\}$ | 5 | nonsymmetric |
| \mathfrak{X}_2 | $\{3, 4\}$ | 5 | nonsymmetric |
| \mathfrak{X}_3 | $\{1, 2\}, \{3, 4\}$ | 4 | symmetric |
| \mathfrak{X}_4 | $\{3, 4, 6\}$ | 4 | nonsymmetric |
| \mathfrak{X}_5 | $\{1, 2\}, \{3, 4\}, \{5, 6\}$ | 3 | symmetric |
| \mathfrak{X}_6 | $\{1, 2, 3, 4\}$ | 3 | symmetric |
| \mathfrak{X}_7 | $\{1, 3\}, \{2, 4\}, \{5, 6\}$ | 3 | nonsymmetric |
| \mathfrak{X}_8 | $\{1, 4\}, \{2, 3\}, \{5, 6\}$ | 3 | nonsymmetric |

Table: Fusion schemes of \mathfrak{X}

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Complex Hadamard matrices and association schemes

$\mathfrak{X} = (X, \{R_i\}_{i=0}^3)$: a 3-class commutative nonsymmetric association scheme with the first eigenmatrix

$$\begin{pmatrix} 1 & \frac{k_1}{2} & \frac{k_1}{2} & k_2 \\ 1 & \frac{1}{2}(r + bi) & \frac{1}{2}(r - bi) & -(r + 1) \\ 1 & \frac{1}{2}(r - bi) & \frac{1}{2}(r + bi) & -(r + 1) \\ 1 & \frac{s}{2} & \frac{s}{2} & -(s + 1) \end{pmatrix}, \quad (4)$$

where

- k_1 is an even positive integer, $k_2 \in \mathbb{Z}$,
- $r, s \in \mathbb{Z}$,
- $b \in \mathbb{R}$ and $b > 0$,
- $i^2 = -1$.

$\mathfrak{A} = \langle A_0, A_1, A_2, A_3 \rangle$: Bose–Mesner algebra of \mathfrak{X} which is the linear span of the adjacency matrices A_0, A_1, A_2, A_3 of \mathfrak{X} , where $A_1^\top = A_2, A_3$ symmetric.

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Complex Hadamard matrices and association schemes

S. Y. Song showed the following.

Lemma 3 (S. Y. Song, 1995)

For the matrix (4), one of the following holds.

- (i) $(r, s, b^2) = (0, -(k_2 + 1), \frac{k_1(k_2+1)}{k_2})$, $m_1 = \frac{(k_1+k_2+1)k_2}{2(k_2+1)}$,
- (ii) $(r, s, b^2) = (-(k_2 + 1), 0, (k_2 + 1)(k_1 + k_2 + 1))$, $m_1 = \frac{k_1}{2(k_2+1)}$,
- (iii) $(r, s, b^2) = (-1, k_1, k_1 + 1)$, $m_1 = \frac{(k_1+k_2+1)k_1}{k_1+1}$.

In Lemma 3,

- (i) and (ii) are nonsymmetric fissions of a complete multipartite graph,
- (i) is self-dual, and (ii) is non self-dual.
- (iii) is a nonsymmetric fission of a disjoint union of complete graphs.

w_1, w_2, w_3 : complex numbers of absolute value 1.

We assume that $w_1 \neq w_2$, and set

$$\textcolor{red}{W} = A_0 + w_1 A_1 + w_2 A_2 + w_3 A_3 \in \mathfrak{A}. \quad (5)$$

Theorem 4 (A. Munemasa and T. I.)

The matrix (5) is a complex Hadamard matrix \iff
 $(k_1, k_2, r, s, b) = (2a(2a-1)c, 2a-1, 0, -2a, 2a\sqrt{c})$ for some
 positive integers a, c , and one of the following holds.

(i) $c = 1$, and

(a) $(w_1, w_2, w_3) = (w, -w, 1)$ with $|w| = 1$,

(b) $(w_1, w_2, w_3) = (w^\pm, w^\mp, w^\pm w^\mp)$, where

$$w^\pm = \frac{-(a-1) - ai \pm ((2a-1)i - 1)\zeta \sqrt{a(a-1)}}{2a^2 - 2a + 1},$$

ζ is a primitive 8-th root of unity, and $i = \zeta^2$,

(c) $a = 2$, $(w_1, w_2, w_3) = (\frac{3 \pm 4i}{5}, -1, \frac{-3 \mp 4i}{5}), (-1, \frac{3 \pm 4i}{5}, \frac{-3 \mp 4i}{5})$,

(ii) $a = 1$, $c = 3$, and

(d) $(w_1, w_2, w_3) = (\frac{1 \pm 2\sqrt{2}i}{3}, -1, 1), (-1, \frac{1 \pm 2\sqrt{2}i}{3}, 1)$,

(e) $(w_1, w_2, w_3) = (\pm i, -1, \mp i), (-1, \pm i, \mp i)$.